

# Superintegrable Deformations of the Smorodinsky–Winternitz Hamiltonian<sup>1</sup>

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**Abstract.** A constructive procedure to obtain superintegrable deformations of the classical Smorodinsky–Winternitz Hamiltonian by using quantum deformations of its underlying Poisson  $sl(2)$  coalgebra symmetry is introduced. Through this example, the general connection between coalgebra symmetry and quasi-maximal superintegrability is analysed. The notion of comodule algebra symmetry is also shown to be applicable in order to construct new integrable deformations of certain Smorodinsky–Winternitz systems.

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## 1. Introduction

The aim of this work is to review a (co)algebraic approach to the superintegrability properties of the classical Smorodinsky–Winternitz (SW) Hamiltonian [12, 18]. As we shall see, the main consequence of making explicit such  $sl(2)$  Poisson coalgebra symmetry is the possibility of constructing superintegrable deformations of the SW Hamiltonian by making use of quantum algebra deformations of  $sl(2)$ . We would also like to emphasize that such deformation procedure is rather general and can be applied to other superintegrable Hamiltonians [7].

In the next Section we recall the essentials of coalgebra symmetry [5] and also the intrinsic superintegrability properties of the associated Hamiltonian systems [7]. Section 3 is devoted to the description of the coalgebra symmetry of the SW system [1], and a set of  $(2N - 2)$  functionally independent constants of the motion (including the Hamiltonian) is deduced by making use of the  $sl(2)$  coalgebra. The non-standard deformation of  $sl(2)$  [10, 16, 19] is then used (Section 4) in order to construct a family of integrable deformations of the SW Hamiltonian with a common set of  $(2N - 2)$  functionally independent deformed integrals of the motion. In Section 5, one of these deformations is shown to be of the Stäckel type [17], and a new set of  $(N - 1)$  integrals related with this separability property is obtained [1]. The notion of comodule algebra symmetry [4] is introduced in Section 6, and it is shown that some specific SW systems have such a new type of dynamical symmetry. Once again, this symmetry enables us to construct a new integrable (but perhaps non-superintegrable) deformation of the SW Hamiltonian. Finally, some remarks and open problems are briefly commented.

## 2. Coalgebra symmetry and superintegrability

We recall that a coalgebra  $(A, \Delta)$  is a (unital, associative) algebra  $A$  endowed with a coproduct map [9, 13]:

$$(2.1) \quad \Delta : A \rightarrow A \otimes A,$$

which is coassociative

$$(2.2) \quad (\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta,$$

*i.e.*, the following diagram is a commutative one:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \Delta \downarrow & & \Delta \otimes id \downarrow \\ A \otimes A & \xrightarrow{id \otimes \Delta} & A \otimes A \otimes A \end{array}$$

This “two-fold way” for the definition of the objects on  $A \otimes A \otimes A$  will be essential as far as superintegrability is concerned. Note that, in addition,  $\Delta$  has to be an algebra homomorphism from  $A$  to  $A \otimes A$ :

$$(2.3) \quad \Delta(ab) = \Delta(a) \Delta(b), \quad \forall a, b \in A.$$

Moreover, if  $A$  is a Poisson algebra and

$$(2.4) \quad \Delta(\{a, b\}_A) = \{\Delta(a), \Delta(b)\}_{A \otimes A}, \quad \forall a, b \in A,$$

we shall say that  $(A, \Delta)$  is a Poisson coalgebra, which will be the relevant object for the construction [5] of classical integrable systems that is summarized in the sequel (see [1,2,6,14,15] for different applications to classical and quantum systems).

Let  $(A, \Delta)$  be a Poisson coalgebra with generators  $X_i$  ( $i = 1, \dots, l$ ), Casimir function  $\mathcal{C}(X_1, \dots, X_l)$  and coassociative coproduct  $\Delta \equiv \Delta^{(2)}$  which is a Poisson map with respect to the Poisson bracket on  $A \otimes A$  given by:

$$(2.5) \quad \{X_i \otimes X_j, X_r \otimes X_s\}_{A \otimes A} = \{X_i, X_r\}_A \otimes X_j X_s + X_i X_r \otimes \{X_j, X_s\}_A.$$

The  $m$ -th coproduct map  $\Delta_L^{(m)} : A \rightarrow A \otimes A \otimes \dots^{(m)} \otimes A$  can be defined by applying recursively the coproduct  $\Delta^{(2)}$  in the form

$$(2.6) \quad \Delta_L^{(m)} := (id \otimes id \otimes \dots^{(m-2)} \otimes id \otimes \Delta^{(2)}) \circ \Delta_L^{(m-1)}.$$

Such an induction ensures that  $\Delta_L^{(m)}$  is also a Poisson map. As a consequence of the definition of  $\Delta_L^{(m)}$ , for any smooth function  $\mathcal{H}(X_1, \dots, X_l)$  we can define a  $N$ -sites Hamiltonian as the  $N$ -th coproduct of  $\mathcal{H}$ :

$$(2.7) \quad H^{(N)} := \Delta_L^{(N)}(\mathcal{H}(X_1, \dots, X_l)) = \mathcal{H}(\Delta_L^{(N)}(X_1), \dots, \Delta_L^{(N)}(X_l)).$$

By construction, it can be proven that the  $(N-1)$  functions given by  $(m = 2, \dots, N)$

$$(2.8) \quad C^{(m)} := \Delta_L^{(m)}(\mathcal{C}(X_1, \dots, X_l)) = \mathcal{C}(\Delta_L^{(m)}(X_1), \dots, \Delta_L^{(m)}(X_l)),$$

Poisson-commute with the Hamiltonian:

$$(2.9) \quad \{C^{(m)}, H^{(N)}\}_{A \otimes A \otimes \dots^{(N)} \otimes A} = 0, \quad m = 2, \dots, N.$$

Moreover, all these integrals of the motion are mutually in involution:

$$(2.10) \quad \{C^{(m)}, C^{(n)}\}_{A \otimes A \otimes \dots^{(N)} \otimes A} = 0, \quad m, n = 2, \dots, N.$$

When the Hamiltonians  $\mathcal{H}$  are defined on Poisson–Lie algebras, the coproduct is “primitive”:  $\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i$ . However, the Poisson analogues of quantum algebras and groups [9, 13] are also (deformed) coalgebras  $(A_z, \Delta_z)$  (where  $z$  is the deformation parameter). Consequently, any function of the generators of a given “quantum” Poisson algebra (with Casimir element  $C_z$ ) will provide, under an appropriate (deformed) symplectic representation, an integrable deformation of the Hamiltonian defined on the coalgebra  $(A, \Delta)$ .

### 2.1. Coalgebras and quasi-maximal superintegrability

Instead of (2.6), another recursion relation for the  $m$ -th coproduct map can be defined:

$$(2.11) \quad \Delta_R^{(m)} := (\Delta^{(2)} \otimes id \otimes \dots^{m-2} \otimes id) \circ \Delta_R^{(m-1)}.$$

Due to the coassociativity property of the coproduct, this new expression will provide exactly the same expressions for the  $N$ -th coproduct of any generator [5]. However, if we label from 1 to  $N$  the sites of the chain of  $N$  copies of  $A$ , lower dimensional coproducts  $\Delta^{(m)}$  (with  $m < N$ ) will be “different” in the sense that  $\Delta_L^{(m)}$  will contain objects living on the tensor product space  $1 \otimes 2 \otimes \dots \otimes m$ , whilst  $\Delta_R^{(m)}$  will be defined on the sites  $(N - m + 1) \otimes (N - m) \otimes \dots \otimes N$ . Therefore, for rank-one coalgebras the coalgebra symmetry of a given Hamiltonian gives rise to two sets of  $(N - 1)$  integrals of the motion that Poisson-commute with  $H^{(N)}$  [7]:

- A set of “left” integrals  $\{C^{(m)} = \Delta_L^{(m)}(\mathcal{C}), m = 2, \dots, N\}$ :

$$\begin{array}{lll} C^{(2)} \equiv \Delta_L^{(2)}(\mathcal{C}) & \text{which is defined on the space} & 1 \otimes 2 \\ C^{(3)} \equiv \Delta_L^{(3)}(\mathcal{C}) & \text{”} & 1 \otimes 2 \otimes 3 \\ \vdots & & \vdots \\ C^{(N)} \equiv \Delta_L^{(N)}(\mathcal{C}) & \text{”} & 1 \otimes 2 \otimes \dots \otimes N \end{array}$$

- A set of “right” integrals  $\{I^{(m)} = \Delta_R^{(m)}(\mathcal{C}), m = 2, \dots, N\}$ :

$$\begin{array}{lll} I^{(2)} \equiv \Delta_R^{(2)}(\mathcal{C}) & \text{on the space} & (N - 1) \otimes N \\ I^{(3)} \equiv \Delta_R^{(3)}(\mathcal{C}) & \text{”} & (N - 2) \otimes (N - 1) \otimes N \\ \vdots & & \vdots \\ I^{(N)} \equiv \Delta_R^{(N)}(\mathcal{C}) & \text{”} & 1 \otimes \dots \otimes (N - 2) \otimes (N - 1) \otimes N \end{array}$$

Note that  $C^{(N)} \equiv I^{(N)}$ . Thus, if all these integrals are functionally independent, we obtain an explicit construction of “quasi-maximally superintegrable systems”, since the coalgebra generates a set of  $(2N - 2)$  functions in involution

$$(2.12) \quad \{H^{(N)}, C^{(2)}, \dots, C^{(N-1)}, C^{(N)} \equiv I^{(N)}, I^{(N-1)}, \dots, I^{(2)}\}.$$

We remark that, in some cases, one more independent integral could exist (leading to a maximally superintegrable system), but such remaining constant of the motion cannot be deduced from the coalgebra symmetry.

### 3. Coalgebra symmetry of the SW Hamiltonian

Let us consider the  $sl(2)$  Poisson coalgebra [1, 16]:

$$(3.1) \quad \begin{aligned} \{J_3, J_+\} &= 2J_+, & \{J_3, J_-\} &= -2J_-, & \{J_-, J_+\} &= 4J_3, \\ \Delta(J_i) &= 1 \otimes J_i + J_i \otimes 1, & i &= +, -, 3, \end{aligned}$$

with Casimir function  $\mathcal{C} = J_3^2 - J_- J_+$ . A one-particle symplectic realization of this coalgebra is given by

$$(3.2) \quad D(J_-) = q_1^2, \quad D(J_+) = p_1^2 + \frac{b_1}{q_1^2}, \quad D(J_3) = q_1 p_1,$$

where  $\{q_1, p_1\} = 1$ . Note that, under this realization,  $D(\mathcal{C}) = -b_1$ .

If we consider the following Hamiltonian function:

$$(3.3) \quad \mathcal{H} = J_+ + \omega^2 J_-,$$

its one-particle realization is just

$$(3.4) \quad H^{(1)} = D(\mathcal{H}) = p_1^2 + \omega^2 q_1^2 + \frac{b_1}{q_1^2}.$$

The 2-particle realization of the coalgebra is obtained through the coproduct:

$$(3.5) \quad \begin{aligned} (D \otimes D)(\Delta^{(2)}(J_-)) &= f_-^{(2)} = q_1^2 + q_2^2, \\ (D \otimes D)(\Delta^{(2)}(J_+)) &= f_+^{(2)} = p_1^2 + p_2^2 + \frac{b_1}{q_1^2} + \frac{b_2}{q_2^2}, \\ (D \otimes D)(\Delta^{(2)}(J_3)) &= f_3^{(2)} = q_1 p_1 + q_2 p_2. \end{aligned}$$

Hence the associated 2-particle Hamiltonian is

$$(3.6) \quad H^{(2)} = (D \otimes D)(\Delta^{(2)}(\mathcal{H})) = \sum_{i=1}^2 \left( p_i^2 + \omega^2 q_i^2 + \frac{b_i}{q_i^2} \right),$$

which is just the  $N = 2$  SW Hamiltonian. A (both left and right) constant of the motion for  $H^{(2)}$  is given by the coproduct of the Casimir:

$$(3.7) \quad C^{(2)} = (D \otimes D)(\Delta^{(2)}(\mathcal{C})) = -(q_1 p_2 - q_2 p_1)^2 - \left( b_1 \frac{q_2^2}{q_1^2} + b_2 \frac{q_1^2}{q_2^2} \right) - \sum_{i=1}^2 b_i.$$

In general, the  $N$ -particle realization is obtained by applying the  $\Delta^{(N)}$  map:

$$\begin{aligned}
 (D \otimes D \otimes \dots^N \otimes D)(\Delta_L^{(N)}(J_-)) &= f_-^{(N)} = \sum_{i=1}^N q_i^2, \\
 (3.8) \quad (D \otimes D \otimes \dots^N \otimes D)(\Delta_L^{(N)}(J_+)) &= f_+^{(N)} = \sum_{i=1}^N \left( p_i^2 + \frac{b_i}{q_i^2} \right), \\
 (D \otimes D \otimes \dots^N \otimes D)(\Delta_L^{(N)}(J_3)) &= f_3^{(N)} = \sum_{i=1}^N q_i p_i,
 \end{aligned}$$

and the  $N$ -particle Hamiltonian given by the coalgebra is just the SW system:

$$(3.9) \quad H^{(N)} = (D \otimes D \otimes \dots^N \otimes D)(\Delta_L^{(N)}(\mathcal{H})) = \sum_{i=1}^N \left( p_i^2 + \omega^2 q_i^2 + \frac{b_i}{q_i^2} \right).$$

The first set of  $(N - 1)$  (left) constants of the motion in involution turns out to be  $(m = 2, \dots, N)$ :

$$(3.10) \quad C^{(m)} = (D \otimes D \otimes \dots^m \otimes D)(\Delta_L^{(m)}(\mathcal{C})) = - \sum_{i < j}^m I_{ij} - \sum_{i=1}^m b_i,$$

where

$$(3.11) \quad I_{ij} = (q_i p_j - q_j p_i)^2 + \left( b_i \frac{q_j^2}{q_i^2} + b_j \frac{q_i^2}{q_j^2} \right).$$

In this way, the complete integrability of the SW Hamiltonian is extracted from the coalgebra symmetry of the model [1].

### 3.1. Coalgebraic superintegrability

Further to the integrability, the coalgebra symmetry also underlies the superintegrability of the SW Hamiltonian since besides the “left integrals”  $C^{(m)}$  (3.10), there exists a set of “right” ones  $I^{(m)}$  given by  $(m = 2, \dots, N)$ :

$$(3.12) \quad I^{(m)} = (D \otimes D \otimes \dots^m \otimes D)(\Delta_R^{(m)}(\mathcal{C})) = - \sum_{N-m+1 \leq i < j}^N I_{ij} - \sum_{i=N-m+1}^N b_i.$$

The functional independence of all these integrals follows from the properties of their  $I_{ij}$  building blocks. Let us firstly consider the  $N = 3$  integrals:

$$\begin{aligned}
 C^{(2)} &= -I_{12} - (b_1 + b_2), & I^{(2)} &= -I_{23} - (b_2 + b_3), \\
 C^{(3)} \equiv I^{(3)} &= -I_{12} - I_{13} - I_{23} - (b_1 + b_2 + b_3),
 \end{aligned}$$

which are functionally independent, since  $C^{(3)}$  contains the  $I_{13}$  term. Similarly for the  $N = 4$  case, where the integrals coming from the coalgebra read

$$\begin{aligned} C^{(2)} &= -I_{12} - (b_1 + b_2), & I^{(2)} &= -I_{34} - (b_3 + b_4), \\ C^{(3)} &= -I_{12} - I_{13} - I_{23} - (b_1 + b_2 + b_3), \\ I^{(3)} &= -I_{23} - I_{24} - I_{34} - (b_2 + b_3 + b_4), \\ C^{(4)} &\equiv I^{(4)} = -I_{12} - I_{13} - I_{14} - I_{23} - I_{24} - I_{34} - (b_1 + b_2 + b_3 + b_4). \end{aligned}$$

Once again, the fact that  $I_{14}$  does appear within  $C^{(4)}$  implies the functional independence of the full set of integrals. In the  $N$ -dimensional case, by following the same construction, the functional independence is proven by considering that  $C^{(N)} \equiv I^{(N)}$  is the only integral that contains the  $I_{1N}$  term. Since the  $N$ -dimensional SW Hamiltonian is, by construction, functionally independent of the  $C^{(N)}$  integral, the quasi-maximal superintegrability of the SW Hamiltonian is proven. Finally, in this particular (separable) case we can take any of the one-particle SW Hamiltonians as the remaining independent integral leading to the full maximal superintegrability of the system.

Furthermore, we stress that a much more general family of coalgebra-symmetric quasi-maximally superintegrable Hamiltonians than (3.9) can also be defined [1]. For instance, let us consider the Hamiltonian function

$$(3.13) \quad \mathcal{H} = J_+ + \mathcal{F}(J_-),$$

where  $\mathcal{F}(J_-)$  is an arbitrary smooth function of  $J_-$ . By construction, any  $N$ -particle Hamiltonian of the form

$$(3.14) \quad H^{(N)} = f_+^{(N)} + \mathcal{F}(f_-^{(N)}) = \sum_{i=1}^N \left( p_i^2 + \frac{b_i}{q_i^2} \right) + \mathcal{F} \left( \sum_{i=1}^N q_i^2 \right),$$

is completely integrable (moreover, quasi-maximally superintegrable), and its constants of the motion are the previous sets  $C^{(m)}$  and  $I^{(m)}$ . Note that in the case  $N = 3$ , this system is just one of the superintegrable potentials given by Evans [11].

#### 4. A superintegrable deformation of the SW Hamiltonian

Now we consider the Poisson analogue [1] of the “non-standard” deformation of  $sl_z(2)$  [16]:

$$(4.1) \quad \begin{aligned} \{J_3, J_+\} &= 2J_+ \cosh zJ_-, \\ \{J_3, J_-\} &= -2 \frac{\sinh zJ_-}{z}, & \{J_-, J_+\} &= 4J_3. \end{aligned}$$

A deformed Casimir function for  $sl_z(2)$  is found to be:

$$(4.2) \quad \mathcal{C}_z = J_3^2 - \frac{\sinh zJ_-}{z} J_+.$$

The deformed coproduct map  $\Delta_z : sl_z(2) \rightarrow sl_z(2) \otimes sl_z(2)$  is given by:

$$\begin{aligned}
(4.3) \quad \Delta_z(J_-) &= J_- \otimes 1 + 1 \otimes J_-, \\
\Delta_z(J_+) &= J_+ \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_+, \\
\Delta_z(J_3) &= J_3 \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_3.
\end{aligned}$$

Let us mimic the construction performed in Section 3 by taking again the function (3.3) for  $\mathcal{H}$ . A one-particle deformed symplectic realization of  $sl_z(2)$  is:

$$\begin{aligned}
(4.4) \quad D_z(J_-) &= q_1^2, \quad D_z(J_+) = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \frac{zb_1}{\sinh zq_1^2}, \\
D_z(J_3) &= \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1,
\end{aligned}$$

which is characterized by the Casimir function  $C_z^{(1)} = D_z(\mathcal{C}_z) = -b_1$ . The associated one-particle Hamiltonian is just:

$$(4.5) \quad H_z^{(1)} = D_z(\mathcal{H}) = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \frac{zb_1}{\sinh zq_1^2} + \omega^2 q_1^2,$$

and the 2-particle symplectic realization is obtained through  $\Delta_z$ :

$$\begin{aligned}
(4.6) \quad (D_z \otimes D_z)(\Delta_z^{(2)}(J_-)) &= \tilde{f}_-^{(2)} = q_1^2 + q_2^2, \\
(D_z \otimes D_z)(\Delta_z^{(2)}(J_+)) &= \tilde{f}_+^{(2)} = \left( \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \frac{zb_1}{\sinh zq_1^2} \right) e^{zq_2^2} \\
&\quad + \left( \frac{\sinh zq_2^2}{zq_2^2} p_2^2 + \frac{zb_2}{\sinh zq_2^2} \right) e^{-zq_1^2}, \\
(D_z \otimes D_z)(\Delta_z^{(2)}(J_3)) &= \tilde{f}_3^{(2)} = \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1 e^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2} q_2 p_2 e^{-zq_1^2}.
\end{aligned}$$

As a consequence, a deformation of the 2-particle SW Hamiltonian is obtained as  $H_z^{(2)} = (D_z \otimes D_z)(\Delta^{(2)}(\mathcal{H}))$ . Namely,

$$\begin{aligned}
(4.7) \quad H_z^{(2)} &= \left( \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \frac{zb_1}{\sinh zq_1^2} \right) e^{zq_2^2} \\
&\quad + \left( \frac{\sinh zq_2^2}{zq_2^2} p_2^2 + \frac{zb_2}{\sinh zq_2^2} \right) e^{-zq_1^2} + \omega^2 (q_1^2 + q_2^2).
\end{aligned}$$

Note that separability is destroyed under deformation. The corresponding constant of the motion is  $C_z^{(2)} = (D_z \otimes D_z)(\Delta^{(2)}(\mathcal{C}_z))$ . Explicitly,

$$\begin{aligned}
(4.8) \quad C_z^{(2)} &= -\frac{\sinh zq_1^2}{zq_1^2} \frac{\sinh zq_2^2}{zq_2^2} (q_1 p_2 - q_2 p_1)^2 e^{z(q_2^2 - q_1^2)} - b_1 e^{2zq_2^2} - b_2 e^{-2zq_1^2} \\
&\quad - \left( b_1 \frac{\sinh zq_2^2}{\sinh zq_1^2} + b_2 \frac{\sinh zq_1^2}{\sinh zq_2^2} \right) e^{z(q_2^2 - q_1^2)}.
\end{aligned}$$



The generic  $m$ -particle symplectic realization is then obtained through the (either right or left)  $m$ -th deformed coproduct and reads:

$$\begin{aligned}
(D_z \otimes \dots^m \otimes D_z)(\Delta_{z,L}^{(m)}(J_3)) &= \tilde{f}_-^{(m)} = \sum_{i=1}^m q_i^2, \\
(D_z \otimes \dots^m \otimes D_z)(\Delta_{z,L}^{(m)}(J_+)) \\
(4.9) \quad &= \tilde{f}_+^{(m)} = \sum_{i=1}^m \left( \frac{\sinh z q_i^2}{z q_i^2} p_i^2 + \frac{z b_i}{\sinh z q_i^2} \right) \exp \left\{ z K_i^{(m)}(q^2) \right\}, \\
(D_z \otimes \dots^m \otimes D_z)(\Delta_{z,L}^{(m)}(J_-)) &= \tilde{f}_3^{(m)} = \sum_{i=1}^m \frac{\sinh z q_i^2}{z q_i^2} q_i p_i \exp \left\{ z K_i^{(m)}(q^2) \right\},
\end{aligned}$$

where the “long-range” interaction is encoded within the functions

$$\begin{aligned}
K_i^{(m)}(q^2) &= - \sum_{k=1}^{i-1} q_k^2 + \sum_{l=i+1}^m q_l^2, \\
(4.10) \quad K_{ij}^{(m)}(q^2) &= K_i^{(m)}(q^2) + K_j^{(m)}(q^2) \\
&= -2 \sum_{k=1}^{i-1} q_k^2 - q_i^2 + q_j^2 + 2 \sum_{l=j+1}^m q_l^2, \quad (i < j).
\end{aligned}$$

In this way, the  $N$ -particle deformed SW Hamiltonian is defined as:

$$\begin{aligned}
H_z^{(N)} &= \tilde{f}_+^{(N)} + \omega^2 \tilde{f}_-^{(N)} \\
(4.11) \quad &= \sum_{i=1}^N \left( \frac{\sinh z q_i^2}{z q_i^2} p_i^2 + \frac{z b_i}{\sinh z q_i^2} \right) \exp \left\{ z K_i^{(N)}(q^2) \right\} + \omega^2 \sum_{i=1}^N q_i^2.
\end{aligned}$$

And the (left) constants of the motion in involution with  $H_z^{(N)}$  are:

$$\begin{aligned}
C_z^{(m)} &= - \sum_{i < j}^m \frac{\sinh z q_i^2}{z q_i^2} \frac{\sinh z q_j^2}{z q_j^2} (q_i p_j - q_j p_i)^2 \exp \left\{ z K_{ij}^{(m)}(q^2) \right\} \\
(4.12) \quad &- \sum_{i < j}^m \left( b_i \frac{\sinh z q_j^2}{\sinh z q_i^2} + b_j \frac{\sinh z q_i^2}{\sinh z q_j^2} \right) \exp \left\{ z K_{ij}^{(m)}(q^2) \right\} \\
&- \sum_{i=1}^m b_i \exp \left\{ 2 z K_i^{(m)}(q^2) \right\}.
\end{aligned}$$

These deformed integrals can also be written as

$$(4.13) \quad C_z^{(m)} = - \sum_{i < j}^m I_{ij}^z \exp \left\{ z K_{ij}^{(m)}(q^2) \right\} - \sum_{i=1}^m b_i \exp \left\{ 2 z K_i^{(m)}(q^2) \right\},$$

where we have defined the following analogues of the  $I_{ij}$  symbols (3.11):

$$(4.14) \quad I_{ij}^z = \frac{\sinh zq_i^2}{zq_i^2} \frac{\sinh zq_j^2}{zq_j^2} (q_i p_j - q_j p_i)^2 + \left( b_i \frac{\sinh zq_j^2}{\sinh zq_i^2} + b_j \frac{\sinh zq_i^2}{\sinh zq_j^2} \right).$$

#### 4.1. Coalgebraic superintegrability of the deformation

By following the very same procedure as in the non-deformed case, the set of deformed right integrals  $I_z^{(m)}$  can be easily constructed and reads

$$(4.15) \quad I_z^{(m)} = - \sum_{N-m+1 \leq i < j}^N I_{ij}^z \exp \left\{ z R_{ij}^{(m)}(q^2) \right\} - \sum_{i=N-m+1}^N b_i \exp \left\{ 2z R_i^{(m)}(q^2) \right\},$$

where the “long-range” interaction  $R$ -functions, similar to (4.10), are defined by

$$(4.16) \quad \begin{aligned} R_i^{(m)}(q^2) &= - \sum_{p=N-m+1}^{i-1} q_p^2 + \sum_{l=i+1}^N q_l^2, \\ R_{ij}^{(m)}(q^2) &= R_i^{(m)}(q^2) + R_j^{(m)}(q^2) \\ &= -2 \sum_{p=N-m+1}^{i-1} q_p^2 - q_i^2 + q_j^2 + 2 \sum_{l=j+1}^N q_l^2, \quad (i < j). \end{aligned}$$

The functional independence of the left and right deformed integrals follows from the fact that they are analytic in the deformation parameter  $z$ :

$$C_z^{(m)} = C^{(m)} + o[z], \quad I_z^{(m)} = I^{(m)} + o[z].$$

As they are functionally independent at  $z = 0$ , they will be so in the whole complex  $z$ -plane, up to isolated points.

Let us explicitly write such integrals in the  $N = 3$  case:

$$\begin{aligned} C_z^{(2)} &= -I_{12}^z \exp \left\{ z K_{12}^{(2)}(q^2) \right\} - \sum_{i=1}^2 b_i \exp \left\{ 2z K_i^{(2)}(q^2) \right\}, \\ C_z^{(3)} \equiv I_z^{(3)} &= -I_{12}^z \exp \left\{ z K_{12}^{(3)}(q^2) \right\} - I_{13}^z \exp \left\{ z K_{13}^{(3)}(q^2) \right\} \\ &\quad - I_{23}^z \exp \left\{ z K_{23}^{(3)}(q^2) \right\} - \sum_{i=1}^3 b_i \exp \left\{ 2z K_i^{(3)}(q^2) \right\} \\ I_z^{(2)} &= -I_{23}^z \exp \left\{ z R_{23}^{(2)}(q^2) \right\} - \sum_{i=2}^3 b_i \exp \left\{ 2z R_i^{(2)}(q^2) \right\}, \end{aligned}$$

where the  $K$  and  $R$ -functions involved in the previous expressions read

$$\begin{aligned}
K_1^{(2)}(q^2) &= q_2^2, & K_2^{(2)}(q^2) &= -q_1^2, & K_{12}^{(2)}(q^2) &= -q_1^2 + q_2^2, \\
K_1^{(3)}(q^2) &= q_2^2 + q_3^2, & K_{12}^{(3)}(q^2) &= -q_1^2 + q_2^2 + 2q_3^2, \\
K_2^{(3)}(q^2) &= -q_1^2 + q_3^2, & K_{13}^{(3)}(q^2) &= -q_1^2 + q_3^2, \\
K_3^{(3)}(q^2) &= -q_1^2 - q_2^2, & K_{23}^{(3)}(q^2) &= -2q_1^2 - q_2^2 + q_3^2, \\
R_2^{(2)}(q^2) &= q_3^2, & R_3^{(2)}(q^2) &= -q_2^2, & R_{23}^{(2)}(q^2) &= -q_2^2 + q_3^2.
\end{aligned}$$

Moreover, the following family of  $N$ -dimensional Hamiltonian systems is also quasi-maximally superintegrable:

$$(4.17) \quad \mathcal{H} = J_+ + \mathcal{F}(J_-),$$

where  $\mathcal{F}(J_-)$  is an arbitrary smooth function of  $J_-$ . Explicitly,

$$\begin{aligned}
(4.18) \quad H_z^{(N)} &= \tilde{f}_+^{(N)} + \mathcal{F}(\tilde{f}_-^{(N)}) \\
&= \sum_{i=1}^N \left( \frac{\sinh z q_i^2}{z q_i^2} p_i^2 + \frac{z b_i}{\sinh z q_i^2} \right) \exp \left\{ z K_i^{(N)}(q^2) \right\} + \mathcal{F} \left( \sum_{i=1}^N q_i^2 \right),
\end{aligned}$$

will Poisson-commute with all the  $C_z^{(m)}$  and  $I_z^{(m)}$ .

## 5. A deformation of Stäckel type

It is obvious that the SW Hamiltonian (3.9) is a Liouville system, and another possible set of integrals of motion in involution is given by

$$(5.1) \quad M_i = p_i^2 + \omega^2 q_i^2 + \frac{b_i}{q_i^2} - \frac{H^{(N)}}{N}, \quad i = 1, \dots, N,$$

where  $\sum_{i=1}^N M_i = 0$ . In order to get the maximal superintegrability of the non-deformed SW Hamiltonian we can take any of these integrals in order to complete, in a functionally independent way, the  $C^{(m)}$  and  $I^{(m)}$  sets of constants of the motion. On the contrary, in the deformed case (4.11), the separability is broken due to the long-range interaction introduced by the deformation. However, since the coalgebra construction allows for an infinite family of deformed Hamiltonians, and all of them Poisson-commute with the same  $C^{(m)}$  and  $I^{(m)}$  sets, it could happen that another choice of the dynamical Hamiltonian could fulfil the separability conditions.

This is the case if we consider the Hamiltonian function [1]:

$$(5.2) \quad \mathcal{H} = J_+ e^{z J_-} + \omega^2 \left( \frac{e^{2z J_-} - 1}{2z} \right).$$

By introducing the  $N$ -th particle symplectic realization (4.9) we obtain

$$(5.3) \quad H_z^{(N)} = \sum_{i=1}^N \frac{\sinh z q_i^2}{z q_i^2} e^{z q_i^2} \exp \left\{ 2z \sum_{k=i+1}^N q_k^2 \right\} \left( p_i^2 + b_i \left( \frac{z q_i}{\sinh z q_i^2} \right)^2 \right) + \omega^2 \left( \frac{\exp \left\{ 2z \sum_{j=1}^N q_j^2 \right\} - 1}{2z} \right),$$

which has the form of a Stäckel system

$$(5.4) \quad H_z^{(N)} = \sum_{i=1}^N a_i(q_1, \dots, q_N) \left( \frac{1}{2} p_i^2 + U_i(q_i) \right),$$

provided that

$$(5.5) \quad \begin{aligned} a_i(q_1, \dots, q_N) &= 2 \frac{\sinh z q_i^2}{z q_i^2} e^{z q_i^2} \exp \left\{ 2z \sum_{k=i+1}^N q_k^2 \right\}, \quad i = 1, \dots, N, \\ U_1(q_1) &= \frac{b_1}{2} \left( \frac{z q_1}{\sinh z q_1^2} \right)^2 + \frac{\omega^2}{4z} e^{z q_1^2} \frac{z q_1^2}{\sinh z q_1^2}, \\ U_i(q_i) &= \frac{b_i}{2} \left( \frac{z q_i}{\sinh z q_i^2} \right)^2, \quad i = 2, \dots, N-1, \\ U_N(q_N) &= \frac{b_N}{2} \left( \frac{z q_N}{\sinh z q_N^2} \right)^2 - \frac{\omega^2}{4z} e^{-z q_N^2} \frac{z q_N^2}{\sinh z q_N^2}. \end{aligned}$$

Stäckel's theorem claims that a Hamiltonian (5.4) admits separation of variables in the Hamilton–Jacobi equation if and only if there exists an  $N \times N$  matrix  $B$  with entries  $b_{ij}(q_j)$  such that

$$(5.6) \quad \det B \neq 0, \quad \sum_{j=1}^N b_{ij}(q_j) a_j(q_1, \dots, q_N) = \delta_{i1}.$$

And this is the case for the new deformed Hamiltonian. The non-vanishing entries of  $B$  and its determinant are found to be

$$(5.7) \quad \begin{aligned} b_{1N}(q_N) &= \frac{z q_N^2}{2 \sinh z q_N^2} e^{-z q_N^2}, \quad b_{i \ i-1}(q_{i-1}) = \frac{z q_{i-1}^2}{\sinh z q_{i-1}^2} e^{-z q_{i-1}^2}, \\ b_{ii}(q_i) &= -\frac{z q_i^2}{\sinh z q_i^2} e^{z q_i^2}, \quad i = 2, \dots, N, \\ \det B &= \frac{1}{2} \prod_{i=1}^N \frac{z q_i^2}{\sinh z q_i^2} e^{-z q_i^2}. \end{aligned}$$

As a consequence, Stäckel's theorem gives us a new set of  $N$  functionally independent integrals of motion in involution

$$(5.8) \quad Z_j = \sum_{i=1}^N a_{ij} \left( \frac{1}{2} p_i^2 + U_i(q_i) \right), \quad j = 1, \dots, N,$$

where  $a_{ij}$  are the entries of  $B^{-1}$ . Then  $a_{i1} = a_i$ , so that the first integral  $I_1$  is just the Hamiltonian. In our case, the non-zero functions  $a_{ij}$  turn out to be

$$(5.9) \quad \begin{aligned} a_{i1} &= 2 \frac{\sinh z q_i^2}{z q_i^2} e^{z q_i^2} \exp \left\{ 2z \sum_{k=i+1}^N q_k^2 \right\}, \quad i = 1, \dots, N, \\ a_{ij} &= \frac{\sinh z q_i^2}{z q_i^2} e^{z q_i^2} \exp \left\{ 2z \sum_{k=i+1}^{j-1} q_k^2 \right\}, \quad i = 1, \dots, N, \quad i < j. \end{aligned}$$

The new set of  $N - 1$  conserved quantities is given by ( $j = 2, \dots, N$ ):

$$(5.10) \quad \begin{aligned} Z_j^z &= \sum_{i=1}^{j-1} \frac{\sinh z q_i^2}{2z q_i^2} e^{z q_i^2} \exp \left\{ 2z \sum_{k=i+1}^{j-1} q_k^2 \right\} \left( p_i^2 + b_i \left( \frac{z q_i}{\sinh z q_i^2} \right)^2 \right) \\ &\quad + \frac{\omega^2}{4z} \left( \exp \left\{ 2z \sum_{k=1}^{j-1} q_k^2 \right\} - 1 \right). \end{aligned}$$

Thus, for instance, the function  $Z_2^z$  can be taken as the remaining constant of the motion, which together with the family of “left” and “right” ones prove the maximal superintegrability of the Hamiltonian (5.3).

In the  $z \rightarrow 0$  limit, the integrals (5.10) reduce to

$$(5.11) \quad Z_j^0 = \frac{1}{2} \sum_{i=1}^{j-1} \left( p_i^2 + \frac{b_i}{q_i^2} \right) + \frac{1}{2} \omega^2 \sum_{k=1}^{j-1} q_k^2.$$

## 6. Comodule algebra symmetry

The notion of coproduct can be generalized by introducing the so called “coactions” [13]. A (right) coaction of a Hopf algebra  $(A, \Delta)$  on a vector space  $V$  is a map  $\phi : V \rightarrow V \otimes A$  such that

$$(6.1) \quad (\phi \otimes id) \circ \phi = (id \otimes \Delta) \circ \phi,$$

that is, if the following diagram is commutative:

$$\begin{array}{ccc}
 V & \xrightarrow{\phi} & V \otimes A \\
 \phi \downarrow & & \phi \otimes id \downarrow \\
 V \otimes A & \xrightarrow{id \otimes \Delta} & V \otimes A \otimes A
 \end{array}$$

If  $V$  is an algebra, we shall say that  $V$  is an  $A$ -comodule algebra if the coaction  $\phi$  is a homomorphism on  $V$

$$(6.2) \quad \phi(ab) = \phi(a)\phi(b), \quad \forall a, b \in V.$$

Moreover, if  $V$  is endowed with a Poisson structure and  $A$  is a Poisson-Hopf algebra,  $V$  will also be a Poisson  $A$ -comodule algebra if:

$$(6.3) \quad \phi(\{a, b\}_V) = \{\phi(a), \phi(b)\}_{V \otimes A}, \quad \forall a, b \in V.$$

Note that any Hopf algebra  $A$  is an  $A$ -comodule algebra with respect to  $A$  provided that  $\phi \equiv \Delta$ . The construction of integrable systems by making use of comodule algebras has been recently introduced in [4] by defining recursively the  $N$ -th coaction as a homomorphism that maps  $V$  within  $V \otimes A \otimes \dots^{(N-1)} \otimes A$ . Let  $\{X_1, \dots, X_l\}$  be the generators of  $V$  and let  $C$  be a Casimir function/operator of  $V$ . It can be proven [4] that the Hamiltonian

$$(6.4) \quad H^{(N)} := \phi^{(N)}(\mathcal{H}(X_1, \dots, X_l)) = \mathcal{H}(\phi^{(N)}(X_1), \dots, \phi^{(N)}(X_l)),$$

together with the following iterated (left) coactions of the Casimir are a set of  $N$  functions in involution and functionally independent ( $m = 2, \dots, N$ ):

$$(6.5) \quad C^{(m)} := \phi^{(m)}(C(X_1, \dots, X_l)) = C(\phi^{(m)}(X_1), \dots, \phi^{(m)}(X_l)).$$

We stress that the “right” integrals cannot be defined in this approach, thus the superintegrability of comodule symmetric systems cannot be ensured algebraically, and it has to be analysed in each particular case.

### 6.1. Comodule algebra symmetry of the SW Hamiltonian

Let us now describe an integrable deformation of the  $N = 2$  SW Hamiltonian (3.6) with comodule algebra symmetry. We take as the Poisson-Hopf algebra  $A$  the following Poisson version of a non-standard deformation [3] of the Schrödinger algebra  $h_6^s$  [8]:

$$\begin{aligned}
 (6.6) \quad & \{\mathcal{D}, \mathcal{P}\} = -\mathcal{P}, & \{\mathcal{D}, \mathcal{K}\} = \mathcal{K}, & \{\mathcal{K}, \mathcal{P}\} = \mathcal{M}, & \{\mathcal{M}, \cdot\} = 0, \\
 & \{\mathcal{D}, \mathcal{H}\} = -2\mathcal{H}, & \{\mathcal{D}, \mathcal{C}\} = 2\mathcal{C}, & \{\mathcal{H}, \mathcal{C}\} = \mathcal{D}, & \{\mathcal{H}, \mathcal{P}\} = 0, \\
 & \{\mathcal{P}, \mathcal{C}\} = -\mathcal{K}, & \{\mathcal{K}, \mathcal{H}\} = \mathcal{P}, & \{\mathcal{K}, \mathcal{C}\} = 0,
 \end{aligned}$$

$$\begin{aligned}
\Delta(\mathcal{M}) &= 1 \otimes \mathcal{M} + \mathcal{M} \otimes 1, \\
\Delta(\mathcal{H}) &= 1 \otimes \mathcal{H} + \mathcal{H} \otimes (1 + \sigma \mathcal{P})^2, \\
\Delta(\mathcal{D}) &= 1 \otimes \mathcal{D} + \mathcal{D} \otimes \frac{1}{1 + \sigma \mathcal{P}} - \frac{1}{2} \mathcal{M} \otimes \frac{\sigma \mathcal{P}}{1 + \sigma \mathcal{P}}, \\
\Delta(\mathcal{C}) &= 1 \otimes \mathcal{C} + \mathcal{C} \otimes \frac{1}{(1 + \sigma \mathcal{P})^2} + \sigma \mathcal{D}' \otimes \frac{1}{1 + \sigma \mathcal{P}} \mathcal{K} \\
&\quad + \frac{\sigma^2}{2} (\mathcal{D}')^2 \otimes \frac{\mathcal{M}}{(1 + \sigma \mathcal{P})^2}, \\
\Delta(\mathcal{P}) &= 1 \otimes \mathcal{P} + \mathcal{P} \otimes 1 + \sigma \mathcal{P} \otimes \mathcal{P}, \\
\Delta(\mathcal{K}) &= 1 \otimes \mathcal{K} + \mathcal{K} \otimes \frac{1}{1 + \sigma \mathcal{P}} + \sigma \mathcal{D}' \otimes \frac{\mathcal{M}}{1 + \sigma \mathcal{P}},
\end{aligned}
\tag{6.7}$$

where  $\mathcal{D}' = \mathcal{D} + \frac{1}{2} \mathcal{M}$ .

The Poisson- $gl(2)$  subalgebra generated by  $\{\mathcal{M}, \mathcal{H}, \mathcal{D}, \mathcal{C}\}$  is a Schrödinger comodule algebra  $V$  and the coaction  $\phi^{(2)} : gl(2) \rightarrow gl(2) \otimes h_6^\sigma$  is given by the restriction to the  $gl(2)$  subalgebra of the full coproduct map in  $h_6^\sigma$  [4]:

$$\phi^{(2)}(X) := \Delta(X), \quad X \in \{\mathcal{M}, \mathcal{H}, \mathcal{D}, \mathcal{C}\}.$$

We take the following symplectic realization of  $h_6^\sigma$ :

$$\begin{aligned}
S(\mathcal{C}) &= \frac{q_1^2}{2}, & S(\mathcal{H}) &= \frac{p_1^2}{2}, & S(\mathcal{D}) &= -p_1 q_1, \\
S(\mathcal{M}) &= \lambda_1^2, & S(\mathcal{K}) &= \lambda_1 q_1, & S(\mathcal{P}) &= \lambda_1 p_1,
\end{aligned}
\tag{6.9}$$

and we consider a different symplectic realization for the  $gl(2)$  subalgebra

$$T(\mathcal{C}) = \frac{q_1^2}{2}, \quad T(\mathcal{H}) = \frac{p_1^2}{2} + \frac{b_1}{q_1^2}, \quad T(\mathcal{D}) = -p_1 q_1, \quad T(\mathcal{M}) = \lambda_1^2.$$

If we take as Hamiltonian function  $H = \mathcal{H} + \mathcal{C}$  we find that

$$H_\sigma^{(1)} = T(H) = T(\mathcal{H}) + T(\mathcal{C}) = \frac{p_1^2}{2} + \frac{q_1^2}{2} + \frac{b_1}{q_1^2},$$

is just the undeformed  $N = 1$  SW Hamiltonian (3.4) with  $\omega^2 = 1$ . But the two-particle case provides in a straightforward way a new integrable deformation of the SW Hamiltonian with comodule algebra symmetry:

$$\begin{aligned}
H_\sigma^{(2)} &= (T \otimes S)(\phi^{(2)}(H)) = (T \otimes S)(\phi^{(2)}(\mathcal{H}) + \phi^{(2)}(\mathcal{C})) \\
&= \frac{1}{2}(p_1^2 + p_2^2) + \frac{b_1}{q_1^2} + \frac{q_1^2}{2(1 + \sigma \lambda_2 p_2)^2} + \frac{q_2^2}{2} \\
&\quad + \sigma \lambda_2 \left( 2 \left( \frac{p_1^2}{2} + \frac{b_1}{q_1^2} \right) p_2 + \frac{q_2(\lambda_1^2 - 2q_1 p_1)}{2(1 + \sigma \lambda_2 p_2)} \right) \\
&\quad + \sigma^2 \lambda_2^2 \left( \left( \frac{p_1^2}{2} + \frac{b_1}{q_1^2} \right) p_2^2 + \frac{(\lambda_1^2 - 2q_1 p_1)^2}{8(1 + \sigma \lambda_2 p_2)^2} \right).
\end{aligned}
\tag{6.12}$$

By considering the  $gl(2)$  Casimir function  $C_V = \frac{1}{4} \mathcal{D}^2 - \mathcal{H} \mathcal{C}$ , the constant of the motion in involution with  $H_\sigma^{(2)}$  is obtained:

$$(6.13) \quad C_\sigma^{(2)} = (T \otimes S)(\phi^{(2)}(C_V)) = (T \otimes S)\left(\frac{1}{4} \Delta(\mathcal{D})^2 - \Delta(\mathcal{H})\right).$$

As expected, the limit  $\sigma \rightarrow 0$  of  $C_\sigma^{(2)}$  is just

$$C_0^{(2)} = -\frac{1}{4}(p_2 q_1 - p_1 q_2)^2 - \frac{b_1}{2} \left(1 + \frac{q_2^2}{q_1^2}\right).$$

Further iterations of the coaction map would provide the corresponding integrable deformation in  $N$  dimensions, but in any case the only non-vanishing centrifugal term would be the one that corresponds to  $b_1$ .

We end with some remarks and open problems. Firstly, note that for higher rank coalgebras, we have a set of right and left integrals coming from each of the Casimir functions of the Poisson algebra. In general, these sets could not be functionally independent under an arbitrary symplectic realization and the number of independent integrals coming from the coalgebra has to be fixed for each individual realization. We also mention that subcoalgebras can also be used in order to extract superintegrability properties, as it was pointed out in [2]. Finally, we think that the search for explicit solutions of the deformed SW Hamiltonians and the corresponding deformed Lax formalism are worthy to be considered in the future, as well as the construction and analysis of the quantum mechanical analogues of the deformed SW Hamiltonians here introduced.

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